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Pricing Discrete Barrier Options Under Stochastic Volatility

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Abstract This paper proposes a new approximation method for pricing barrier options with discrete monitoring under stochastic volatility environment. In particular, the integration-by-parts formula and the duality formula in Malliavin calculus are effectively applied in pricing barrier options with discrete monitoring. To the best of our knowledge, this paper is the first one that shows an analytical approximation for pricing discrete barrier options with stochastic volatility models. Furthermore, it provides numerical examples for pricing double barrier call options with discrete monitoring under Heston and λ-SABR models.

Keywords Discrete barrier option · Barrier option · Knock-out option · Double barrier option · Stochastic volatility · CEV model · Heston model · SABR model · λ-SABR model · Asymptotic expansion · Malliavin calculus

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1 Introduction

This paper develops a new approximation formula for pricing discrete barrier options under general stochastic volatility models. In particular, the paper applies the Malliavin calculus to pricing path-dependent derivatives with discrete monitoring under stochastic volatility environment and derives a concrete approximation formula for valuation of barrier options. It is also stressed that our new analytic formula is obtained by an asymptotic expansion around a multi-dimensional log-normal (or Gaussian) distribution, which can be regarded as an extension of existing expansions around one-dimensional Gaussian distributions for approximating derivative prices such as plain-vanilla and average option prices. (For instance, see Takahashi (1999, 2009)). Furthermore, numerical examples for pricing discrete double barrier options under Heston and λ-SABR models are presented.

As a seminal work, Fournié et al. (1999) applied Malliavin calculus to derive efficient Monte Carlo estimators of computing Greeks for path-independent as well as path-dependent options in the Black-Scholes framework. These estimators are sometimes called *Malliavin weights*. Subsequently, a number of papers extended their method. Related to our present work, Siopacha and Teichmann (2011) developed strong and weak Taylor methods for stochastic differential equations. In particular, the weak Taylor expansion is based on the Malliavin's integration by parts on the Wiener space and the expansion coefficients are given by *Malliavin weights*. As an example, they applied the method to a market model of interest rates with stochastic volatility and obtained a *semi*-closed-form approximation of the option prices with expectation including the Malliavin weights; their method depends on the Monte Carlo simulations in order to compute the option prices numerically.

Takahashi and Yamada (2009) gave a perturbation method for stochastic volatility models and pointed out that the approximation terms including the Malliavin weights can be transformed into a finite-dimensional integration through the *duality formula* and obtained *completely* closed-form approximations for density functions and option prices by an asymptotic expansion. Applying both the integration by parts and the *duality formula*, this paper derives a closed-form approximation for prices of barrier options with discrete monitoring as an example. The same method can be used for obtaining closed-form approximations of other derivatives' prices and implied volatilities as well as of the underlying density functions; for instance Takahashi and Yamada (2009) applies the method to deriving expansions of implied volatilities under stochastic volatility models and jump-diffusion models with stochastic volatilities. Also, we remark that there are various approaches for approximations of derivatives' prices, Greeks and heat kernels through certain asymptotic expansions: for instance, there are recent works such as Baudoin (2009), Gatheral et al. (2009) and Ben Arous and Laurence (2009).

As for pricing discrete barrier options, Fusai et al. (2006) provided an analytical solution in the Black-Scholes framework. Recently, using a high-order asymptotic expansion scheme for a plain-vanilla option's value by Takahashi et al. (2009) combined with a static hedging method by Fink (2003), Shiraya et al. (2009) provided an analytic approximation for valuation of barrier options with continuous monitoring under stochastic volatility environment; however, their method cannot be applied to pricing discrete barrier options. Our approximation for the discrete barrier options is made around the log-normal distribution for the Heston-type model and the normal distribution for CEV (Constant Elasticity of Variance) model with general stochastic volatility. To the best of our knowledge, this paper is the first one that derives an analytic (approximation) formula for pricing discrete barrier options with stochastic volatility models. In particular, our result can be regarded as an extension of Fusai et al. (2006).

Moreover, we remark that numerical computations for pricing discrete barrier options under stochastic volatility models typically apply double integrals on the underlying asset and its volatility, which requires computational burden in pricing substantially. On the other hand, our new approximation based on the asymptotic expansion technique is able to price them in a second for our numerical experiments. It implies that our developed method seems useful especially because this type of options is embedded in structured bonds very often where an efficient computational scheme is very desirable.

The organization of the paper is as follows: The next section derives an asymptotic expansion formula for generalized Wiener functionals. Section 3 applies the general formula to pricing path-dependent derivatives with discrete monitoring and provides a concrete approximation formula for valuation of discrete barrier options. Section 4 provides numerical examples for pricing double barrier call options with discrete monitoring under Heston and λ-SABR models. Section 5 concludes. Appendix summarizes Malliavin calculus necessary for this paper.

2 Asymptotic Expansion

2.1 Asymptotic Expansion for Expectation of Generalized Wiener Functionals

The next theorem and corollary present asymptotic expansion formulas for the expectation and the density of generalized Wiener functionals which is a key tool to evaluate the prices of the discrete barrier options under stochastic volatility models. For the definitions, the notations and the proofs, see Takahashi and Yamada (2009). Hereafter, we use the notation $\int T(x)p(x)dx$ for $T \in \mathcal{S}'(\mathbf{R}^n)$ and $p \in \mathcal{S}(\mathbf{R}^n)$ meaning that ${}_{\mathcal{S}'}\langle T, p\rangle_{\mathcal{S}}$.

Theorem 2.1 [Takahashi and Yamada 2009] *Consider a family of smooth Wiener functionals $F^\epsilon = (F_1^\epsilon, \ldots, F_n^\epsilon) \in \mathbf{D}_\infty(\mathcal{W} : \mathbf{R}^n)$ such that F^ϵ has an asymptotic expansion in $\mathbf{D}_\infty$ and satisfies the uniformly non-degenerate condition:*

$$\limsup_{\epsilon \downarrow 0} \|(\det \sigma_{F^\epsilon})^{-1}\|_{L^p} < \infty, \ p < \infty. \tag{2.1}$$

Then, for a Schwartz distribution $T \in S'(\mathbf{R}^n)$, we have an asymptotic expansion in $\mathbf{R}$:

$$E[T(F^{\epsilon})] = \int_{\mathbf{R}^n} T(x) p^{F^0}(x) dx + \sum_{j=1}^{N} \epsilon^j$$
$$\int_{\mathbf{R}^n} T(x) E\left[\sum_k^{(j)} H_{\alpha^{(k)}} \left(F^0, \prod_{l=1}^{k} F^{0,\beta_l}_{\alpha_l} \right) | F^0 = x \right] p^{F^0}(x) dx + O(\epsilon^{N+1}), \tag{2.2}$$

where $F_i^{0,k} := \frac{1}{k!} \frac{d^k}{d\epsilon^k} F_i^{\epsilon}|_{\epsilon=0}$, $k \in \mathbf{N}$, $i = 1, \ldots, n$, $\alpha^{(k)}$ *denotes a multi-index,* $\alpha^{(k)} = (\alpha_1, \ldots, \alpha_k)$ *and*

$$\sum_k^{(j)} \equiv \sum_{k=1}^{j} \sum_{\beta_1 + \cdots + \beta_k = j, \beta_i \geq 1} \sum_{\alpha^{(k)} \in \{1,\ldots,n\}^k} \frac{1}{k!}.$$

Also, Malliavin weight $H_{\alpha^{(k)}}$ *is recursively defined as follows:*

$$H_{\alpha^{(k)}}(F, G) = H_{(\alpha_k)}\left(F, H_{\alpha^{(k-1)}}(F, G) \right),$$

where

$$H_{(l)}(F, G) = D^* \left(\sum_{i=1}^{n} G \gamma_{li}^F D F_i \right).$$

Here, $\gamma^F = \{\gamma_{ij}^F\}_{1 \leq i, j \leq n}$ *denotes the inverse matrix of the Malliavin covariance matrix of* F.

Proof See Takahashi and Yamada (2009). □

Corollary 2.1 *The density* $p^{F^{\epsilon}}(y)$ *is expressed as the following asymptotic expansion with the push-down of Malliavin weights:*

$$p^{F^{\epsilon}}(y) = p^{F^0}(y) + \sum_{j=1}^{m} \epsilon^j E\left[\sum_k^{(j)} H_{\alpha^{(k)}} \left(F^0, \prod_{l=1}^{k} F^{0,\beta_l}_{\alpha_l} \right) | F^0 = y \right] p^{F^0}(y)$$
$$+ O(\epsilon^{m+1}), \tag{2.3}$$

where $p^{F^0}(y)$ *is the density of* F^0.

Proof See Takahashi and Yamada (2009). □

3 Pricing Path-dependent Derivatives with Discrete Monitoring

This section presents an approximation formula for pricing a path-dependent derivative whose payoff is determined by the underlying asset's value at finite number of time points during the contract period, as an application of Theorem 2.1 in the previous section.

3.1 General Results

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space and $(W_{1,t}, W_{2,t})_{t\in[0,T]}$ be a two dimensional Brownian motion with respect to $(\mathcal{F}_t)_{t\in[0,T]}$. We consider the following stochastic volatility model;

$$
\begin{aligned}
dS_t^{(\epsilon)} &= \alpha S_t^{(\epsilon)} dt + V(\sigma_t^{(\epsilon)}, t) S_t^{(\epsilon)} dW_{1,t},\\
d\sigma_t^{(\epsilon)} &= A_0(\sigma_t^{(\epsilon)}, t)dt + \epsilon A_1(\sigma_t^{(\epsilon)}, t)(\rho dW_{1,t} + \sqrt{1-\rho^2} dW_{2,t}),\\
S_0^{(\epsilon)} &= S_0^{(0)} = s,
\end{aligned}
\tag{3.1}
$$

where α is a constant, $\rho \in [-1, 1]$ and $\epsilon \in [0, 1]$. $V, A_0, A_1 : \mathbf{R} \times [0, T] \mapsto \mathbf{R}$ are continuous and C^∞ for each $t \in [0, T]$ with bounded derivatives of any orders in the first argument. Note that $\alpha = r - \delta$, where r and δ are the risk-free rate and continuous dividend rate, respectively. Note also that the model becomes the Black-Scholes model when $\epsilon = 0$.

Under this stochastic volatility model, we consider a derivative whose payoff depends on the underlying asset price S at monitoring time points, $0 = t_0 \leq t_1 \leq \cdots \leq t_N = T$. More specifically, let $\varphi : \mathbf{R}^N \mapsto \mathbf{R}$ be the payoff function of a path-dependent derivative with discrete monitoring. First, we impose the following assumption.

Assumption 3.1 *For all $t \in (0, T]$,*

$$
V(x, t)^2 > 0. \tag{3.2}
$$

Denote $X_{t_i}^{(\epsilon)}$ by the logarithmic process of $S_{t_i}^{(\epsilon)}$;

$$
X_{t_i}^{(\epsilon)} := \log S_{t_i}^{(\epsilon)}, \quad i = 1, \ldots, N.
$$

Then, regarding the valuation of the path-dependent derivative with discrete monitoring, the following theorem is obtained.

Proposition 3.1 *Let $\varphi : \mathbf{R}^N \mapsto \mathbf{R}$ be the payoff function of a path-dependent derivative with discrete monitoring. Then, an asymptotic expansion formula for valuation of the derivative under the stochastic volatility model (3.1) is given by*

$$
\begin{aligned}
&e^{-rT} E[\varphi(S_{t_1}^{(\epsilon)}, \ldots, S_{t_N}^{(\epsilon)})]\\
&\quad = e^{-rT} \int_{\mathbf{R}^N} \varphi(e^{x_1}, \ldots, e^{x_N}) p^{X^{(0)}}(x_1, \ldots, x_N) dx_1 \ldots dx_N\\
&\qquad + \sum_{j=1}^{m} \epsilon^j e^{-rT} \int_{\mathbf{R}^N} \varphi(e^{x_1}, \ldots, e^{x_N}) \sum_{k}^{(j)} E\left[H_{\alpha^{(k)}}\left(X^{(0)}, \prod_{l=1}^{k} X_{\beta_l}^{0,\alpha_l} \right)\right.
\end{aligned}
$$

$$\left. |X^{(0)} = (x_1, \ldots, x_N) \right] p^{X^{(0)}}(x_1, \ldots, x_N) dx_1, \ldots, dx_N$$
$$+ O(\epsilon^{m+1}), \tag{3.3}$$

where $X_k^{0,i} := \frac{1}{k!} \frac{d^k}{d\epsilon^k} X_{t_i}^{(\epsilon)}|_{\epsilon=0}, k \in \mathbf{N}, i = 1, \ldots, N$ *and* $p^{X^0}(x_1, \ldots, x_N)$ *is the density function of* $X^{(0)} = (X_{t_1}^{(0)}, \ldots, X_{t_N}^{(0)})$. *In particular, the first term on the right hand side of (3.32) gives the value of the derivative under the Black-Scholes model.*

Proof The Malliavin covariance matrix $\{\sigma_{X^{(0)}}\}_{ij}$ is given by

$$\sigma_{X^0} = \begin{bmatrix} \langle DX_{t_1}^{(0)}, DX_{t_1}^{(0)} \rangle_H & \langle DX_{t_1}^{(0)}, DX_{t_2}^{(0)} \rangle_H & \cdots & \langle DX_{t_1}^{(0)}, DX_{t_{N-1}}^{(0)} \rangle_H & \langle DX_{t_1}^{(0)}, DX_{t_N}^{(0)} \rangle_H \\ \langle DX_{t_2}^{(0)}, DX_{t_1}^{(0)} \rangle_H & \langle DX_{t_2}^{(0)}, DX_{t_2}^{(0)} \rangle_H & \cdots & \langle DX_{t_2}^{(0)}, DX_{t_{N-1}}^{(0)} \rangle_H & \langle DX_{t_2}^{(0)}, DX_{t_N}^{(0)} \rangle_H \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle DX_{t_{N-1}}^{(0)}, DX_{t_1}^{(0)} \rangle_H & \langle DX_{t_{N-1}}^{(0)}, DX_{t_2}^{(0)} \rangle_H & \cdots & \langle DX_{t_{N-1}}^{(0)}, DX_{t_{N-1}}^{(0)} \rangle_H & \langle DX_{t_{N-1}}^{(0)}, DX_{t_N}^{(0)} \rangle_H \\ \langle DX_{t_N}^{(0)}, DX_{t_1}^{(0)} \rangle_H & \langle DX_{t_N}^{(0)}, DX_{t_2}^{(0)} \rangle_H & \cdots & \langle DX_{t_N}^{(0)}, DX_{t_{N-1}}^{(0)} \rangle_H & \langle DX_{t_N}^{(0)}, DX_{t_N}^{(0)} \rangle_H \end{bmatrix}$$

$$= \begin{bmatrix} \int_0^T V(\sigma_t^{(0)}, t)^2 \mathbf{1}_{t \le t_1} dt & \int_0^T V(\sigma_t^{(0)}, t)^2 \mathbf{1}_{t \le t_1} dt & \cdots & \int_0^T V(\sigma_t^{(0)}, t)^2 \mathbf{1}_{t \le t_1} dt & \int_0^T V(\sigma_t^{(0)}, t)^2 \mathbf{1}_{t \le t_1} dt \\ \int_0^T V(\sigma_t^{(0)}, t)^2 \mathbf{1}_{t \le t_1} dt & \int_0^T V(\sigma_t^{(0)}, t)^2 \mathbf{1}_{t \le t_2} dt & \cdots & \int_0^T V(\sigma_t^{(0)}, t)^2 \mathbf{1}_{t \le t_2} dt & \int_0^T V(\sigma_t^{(0)}, t)^2 \mathbf{1}_{t \le t_2} dt \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \int_0^T V(\sigma_t^{(0)}, t)^2 \mathbf{1}_{t \le t_1} dt & \int_0^T V(\sigma_t^{(0)}, t)^2 \mathbf{1}_{t \le t_2} dt & \cdots & \int_0^T V(\sigma_t^{(0)}, t)^2 \mathbf{1}_{t \le t_{N-1}} dt & \int_0^T V(\sigma_t^{(0)}, t)^2 \mathbf{1}_{t \le t_{N-1}} dt \\ \int_0^T V(\sigma_t^{(0)}, t)^2 \mathbf{1}_{t \le t_1} dt & \int_0^T V(\sigma_t^{(0)}, t)^2 \mathbf{1}_{t \le t_2} dt & \cdots & \int_0^T V(\sigma_t^{(0)}, t)^2 \mathbf{1}_{t \le t_{N-1}} dt & \int_0^T V(\sigma_t^{(0)}, t)^2 \mathbf{1}_{t \le t_N} dt \end{bmatrix}$$

$$= \begin{bmatrix} \int_0^{t_1} V(\sigma_t^{(0)}, t)^2 dt & \int_0^{t_1} V(\sigma_t^{(0)}, t)^2 dt & \cdots & \int_0^{t_1} V(\sigma_t^{(0)}, t)^2 dt & \int_0^{t_1} V(\sigma_t^{(0)}, t)^2 dt \\ \int_0^{t_1} V(\sigma_t^{(0)}, t)^2 dt & \int_0^{t_2} V(\sigma_t^{(0)}, t)^2 dt & \cdots & \int_0^{t_2} V(\sigma_t^{(0)}, t)^2 dt & \int_0^{t_2} V(\sigma_t^{(0)}, t)^2 dt \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \int_0^{t_1} V(\sigma_t^{(0)}, t)^2 dt & \int_0^{t_2} V(\sigma_t^{(0)}, t)^2 dt & \cdots & \int_0^{t_{N-1}} V(\sigma_t^{(0)}, t)^2 dt & \int_0^{t_{N-1}} V(\sigma_t^{(0)}, t)^2 dt \\ \int_0^{t_1} V(\sigma_t^{(0)}, t)^2 dt & \int_0^{t_2} V(\sigma_t^{(0)}, t)^2 dt & \cdots & \int_0^{t_{N-1}} V(\sigma_t^{(0)}, t)^2 dt & \int_0^{t_N} V(\sigma_t^{(0)}, t)^2 dt \end{bmatrix}. \tag{3.4}$$

For $n = 1, \cdots, N$, define

$$\Sigma_n = \begin{bmatrix} \int_0^{t_1} V(\sigma_t^{(0)}, t)^2 dt & \int_0^{t_1} V(\sigma_t^{(0)}, t)^2 dt & \cdots & \int_0^{t_1} V(\sigma_t^{(0)}, t)^2 dt & \int_0^{t_1} V(\sigma_t^{(0)}, t)^2 dt \\ \int_0^{t_1} V(\sigma_t^{(0)}, t)^2 dt & \int_0^{t_2} V(\sigma_t^{(0)}, t)^2 dt & \cdots & \int_0^{t_2} V(\sigma_t^{(0)}, t)^2 dt & \int_0^{t_2} V(\sigma_t^{(0)}, t)^2 dt \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \int_0^{t_1} V(\sigma_t^{(0)}, t)^2 dt & \int_0^{t_2} V(\sigma_t^{(0)}, t)^2 dt & \cdots & \int_0^{t_{n-1}} V(\sigma_t^{(0)}, t)^2 dt & \int_0^{t_{n-1}} V(\sigma_t^{(0)}, t)^2 dt \\ \int_0^{t_1} V(\sigma_t^{(0)}, t)^2 dt & \int_0^{t_2} V(\sigma_t^{(0)}, t)^2 dt & \cdots & \int_0^{t_{n-1}} V(\sigma_t^{(0)}, t)^2 dt & \int_0^{t_n} V(\sigma_t^{(0)}, t)^2 dt \end{bmatrix}. \tag{3.5}$$

The determinant of Σ_n is given by

$$\det \Sigma_n = \Sigma_{(0,1)} \Sigma_{(1,2)} \ldots \Sigma_{(n-1,n)}, \tag{3.6}$$

where

$$\Sigma_{(i-1,i)} = \int_{t_{i-1}}^{t_i} V(\sigma_t^{(0)}, t)^2 dt. \tag{3.7}$$

By Assumption 3.1, each principal minor's determinant of the Malliavin covariance matrix is positive;

$$\det \Sigma_n > 0, \quad n = 1, \ldots, N. \tag{3.8}$$

Then the Malliavin covariance matrix is positive definite. Thus, the uniformly non-degenerate condition is satisfied by the similar argument to Takahashi and Yoshida (2004). For the payoff function $\varphi \in \mathcal{S}'$, Theorem 2.1, especially, the Eq. (2.2) can be applied and hence the following asymptotic expansion formula is obtained:

$$\begin{aligned}
&e^{-rT} E[\varphi(S_{t_1}^{(\epsilon)}, \ldots, S_{t_N}^{(\epsilon)})] \\
&= e^{-rT} \int_{\mathbf{R}^N} \varphi(e^{x_1}, \ldots, e^{x_N}) p^{X^0}(x_1, \ldots, x_N) dx_1 \ldots dx_N \\
&\quad + \sum_{j=1}^{m} \epsilon^j e^{-rT} \int_{\mathbf{R}^N} \varphi(e^{x_1}, \ldots, e^{x_N}) \sum_{k}^{(j)} E\left[H_\alpha \left(X^0, \prod_{l=1}^{k} X_{\beta_l}^{0,\alpha_l} \right) \right. \\
&\quad \left. |X^0 = (x_1, \ldots, x_N) \right] p^{X^0}(x_1, \ldots, x_N) dx_1, \ldots, dx_N \\
&\quad + O(\epsilon^{m+1}).
\end{aligned}$$

□

3.2 Pricing Barrier Options with Discrete Monitoring

This subsection provides an approximation formula for valuation of barrier options with discrete monitoring as a concrete example of the previous subsection. Let $B \subset \mathbf{R}$ be the barrier. For example, $B = [L, \infty)$, $B = (-\infty, H]$ and $[L, H]$ for some constants, $-\infty < L < H < \infty$. Hereafter, the following notations are used:

$$Y_{t_i} := \int_0^{t_i} V(\sigma_t^{(0)}, t) dW_{1t},$$

$$\xi_i := \int_0^{t_i} V(\sigma_t^{(0)}, t)^2 dt - 2\alpha t_i,$$

$$X_{t_i} := \log S_0 + Y_{t_i} - \frac{1}{2}\xi_i,$$

$$v_{1t} := \partial_x V(\sigma_t^{(0)}, t)\sigma_t^{(1)}, \ \partial_x V(\sigma_t^{(0)}, t) := \frac{\partial V(x,t)}{\partial x}|_{x=\sigma_t^{(0)}}$$

$$\sigma_t^{(1)} := \frac{\partial \sigma_t^{(\epsilon)}}{\partial \epsilon}|_{\epsilon=0} = \eta_t \int_0^t \eta_s^{-1} A_1(\sigma_s^{(0)})(\rho dW_{1,s} + \sqrt{1-\rho^2} dW_{2,s}),$$

$$\eta_t := \exp\left\{ \int_0^t \partial_x A_0(\sigma_u^{(0)}, u) du \right\},$$

$$\Psi_i := \int_0^{t_i} v_{1t} dW_{1t} - \int_0^{t_i} V(\sigma_t^{(0)}, t) v_{1t} dt,$$

$$\Psi_{i-1,i} := \int_{t_{i-1}}^{t_i} v_{1t} dW_{1t} - \int_{t_{i-1}}^{t_i} V(\sigma_t^{(0)}, t) v_{1t} dt,$$

$$\Sigma_{(i-1,i)} := \int_{t_{i-1}}^{t_i} V(\sigma_t^{(0)}, t)^2 dt.$$

$Barrier_N^{SV}$ denotes the price at time 0 of a discrete barrier option with strike K and maturity T under the stochastic volatility (3.1). Also, $Barrier_N^{BS}$ denotes the price of this discrete barrier option under the Black-Scholes model;

$$Barrier_N^{BS} = e^{-rT} \int_{\mathbf{R}^N} \psi(y_1, \ldots, y_N) p(y_1, \ldots, y_N) dy_1 \ldots dy_N, \tag{3.9}$$

where

$$\psi(y_1, \ldots, y_N) = \mathbf{1}_{\{se^{y_1 - \frac{1}{2}\xi_1} \in B\}} \cdots \mathbf{1}_{\{se^{y_N - \frac{1}{2}\xi_N} \in B\}} (se^{y_N - \frac{1}{2}\xi_N} - K)^+, \tag{3.10}$$

$$p(y_1, \ldots, y_N) = \frac{1}{(2\pi)^{N/2} \Sigma_{(0,1)}^{1/2} \cdots \Sigma_{(N-1,N)}^{1/2}} e^{-\sum_{i=1}^N \frac{(y_i - y_{i-1})^2}{2\Sigma_{i-1,i}}}, \ (y_0 = 0). \tag{3.11}$$

Then, the following result is obtained. That is, the value of a discrete barrier option under the stochastic volatility is approximated around the value under the Black-Scholes model.

Theorem 3.1 *An asymptotic expansion of $Barrier_N^{SV}$, the price at time 0 of a discrete barrier option with strike K and maturity T under the stochastic volatility (3.1) is given by:*

$$\begin{aligned} Barrier_N^{SV} &= Barrier_N^{BS} \\ &+\epsilon e^{-rT} \int_{\mathbf{R}^N} \psi(y_1,\ldots,y_N)\vartheta(y_1,\ldots,y_N)p(y_1,\ldots,y_N)dy_1\ldots dy_N \\ &+O(\epsilon^2), \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} \vartheta(y_1,\ldots,y_N) = \sum_{k=1}^{N} \zeta_{k-1,k} \Bigg(& \frac{(y_k-y_{k-1})^3}{\Sigma_{k-1,k}^3} - \frac{3(y_k-y_{k-1})}{\Sigma_{k-1,k}} \\ & - \frac{(y_k-y_{k-1})^2}{\Sigma_{k-1,k}^2} + \frac{1}{\Sigma_{k-1,k}} \Bigg) \\ & + \sum_{k=2}^{N}\sum_{l=1}^{k-1} \zeta_{k-1,k}^{(l-1,l)} \left(\frac{y_l-y_{l-1}}{\Sigma_{l-1,l}} \right) \\ & \left(\frac{(y_k-y_{k-1})^2}{\Sigma_{k-1,k}^2} - \frac{1}{\Sigma_{k-1,k}} - \frac{y_k-y_{k-1}}{\Sigma_{k-1,k}} \right), \end{aligned} \tag{3.13}$$

with

$$\zeta_{k-1,k} = \rho \int_{t_{k-1}}^{t_k} \partial V(\sigma_t^{(0)},t)\eta_t V(\sigma_t^{(0)},t) \int_{t_{k-1}}^{t} \eta_s^{-1} A_1(\sigma_s^{(0)},s)V(\sigma_s^{(0)},s)dsdt, \tag{3.14}$$

and

$$\zeta_{k-1,k}^{(l-1,l)} = \rho \int_{t_{k-1}}^{t_k} \partial V(\sigma_t^{(0)},t)\eta_t V(\sigma_t^{(0)},t)dt \int_{t_{l-1}}^{t_l} \eta_s^{-1} A_1(\sigma_s^{(0)},s)V(\sigma_s^{(0)},s)ds. \tag{3.15}$$

Proof

We will evaluate the ϵ's coefficient of the expansion (2.2) in Theorem 2.1 under the current setting.

Let $\varphi : (x_1,\ldots,x_N) \mapsto \mathbf{1}_{e^{x_1}\in B}\ldots\mathbf{1}_{e^{x_N}\in B}(e^{x_N}-K)^+$.

$$\begin{aligned} Barrier_N^{SV} &= e^{-rT}{}_{\mathbf{D}_{-\infty}}\langle \phi(X^{(0)}), \mathbf{1}\rangle_{\mathbf{D}_\infty} + \epsilon e^{-rT} \sum_{i=1}^{N} {}_{\mathbf{D}_{-\infty}}\langle \partial_i\phi(X^{(0)}), \Psi_i\rangle_{\mathbf{D}_\infty} + O(\epsilon^2) \\ &= e^{-rT}{}_{\mathbf{D}_{-\infty}}\langle \phi(X^{(0)}), \mathbf{1}\rangle_{\mathbf{D}_\infty} + \epsilon e^{-rT}{}_{\mathbf{D}_{-\infty}}\langle \phi(X^{(0)}), \pi\rangle_{\mathbf{D}_\infty} + O(\epsilon^2) \\ &= Barrier_N^{BS} + \epsilon e^{-rT}\langle \psi, \vartheta\rangle_{p(y)dy} + O(\epsilon^2) \\ &= Barrier_N^{BS} \end{aligned}$$

$$
+\epsilon e^{-rT}\int_{\mathbf{R}^N}\psi(y_1,\ldots,y_N)\vartheta(y_1,\ldots,y_N)p(y_1,\ldots,y_N)dy_1\ldots dy_N
$$
$$
+O(\epsilon^2), \tag{3.16}
$$

where $\pi = \sum_{i,j=1}^N \pi_{ij} \in \mathbf{D}_\infty$ is the Malliavin weight defined by $\pi_{ij} = D_1^*\left(\Psi_i \gamma_{ij}^{X^{(0)}} D_1 X_j^{(0)}\right)$ and $\vartheta(y_1,\ldots,y_N) = E^{Y=y}[\pi]$ is the push down of the Malliavin weight. The inverse matrix of the Malliavin covariance matrix of $X^{(0)}$ is given by

$$
\gamma^{X^{(0)}} = \begin{bmatrix}
\gamma_{1,1} & \gamma_{1,2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\gamma_{2,1} & \gamma_{2,2} & \gamma_{2,3} & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & \gamma_{3,2} & \gamma_{3,3} & \gamma_{3,4} & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & \gamma_{4,3} & \gamma_{4,4} & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & \cdots & \gamma_{N-3,N-3} & \gamma_{N-3,N-2} & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & \gamma_{N-2,N-3} & \gamma_{N-2,N-2} & \gamma_{N-2,N-1} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & \gamma_{N-1,N-2} & \gamma_{N-1,N-1} & \gamma_{N-1.N} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \gamma_{N,N-1} & \gamma_{N,N}
\end{bmatrix}, \tag{3.17}
$$

where $\gamma_{i,i} = \frac{1}{\Sigma_{(i-1,i)}} + \frac{1}{\Sigma_{(i,i+1)}}, i = 1,\ldots,N-1, \gamma_{i,i+1} = \gamma_{i+1,i} = -\frac{1}{\Sigma_{(i,i+1)}}, i = 1,\ldots,N-1, \gamma_{N,N} = \frac{1}{\Sigma_{(N-1,N)}}$. To make the right-hand side of (3.9) short, let

$$
\Psi_k := \int_0^{t_k} A_t \int_0^t B_s(\rho dW_{1,s} + \sqrt{1-\rho^2}dW_{2,s})dW_{1t}
$$
$$
-\int_0^{t_k} C_t \int_0^t B_s(\rho dW_{1,s} + \sqrt{1-\rho^2}dW_{2,s})dt, \tag{3.18}
$$

where $A_t = \partial_x V(\sigma_t^{(0)}, t)\cdot\eta_t$, $B_s = \eta_s^{-1}A_1(\sigma_s^{(0)}, s)$ and $C_t = V(\sigma_t^{(0)}, t)\cdot\partial_x V(\sigma_t^{(0)}, t)\cdot \eta_t = V(\sigma_t^{(0)}, t)A_t$. Also, Ψ_k is divided into the three parts:

$$
\Psi_k = \Psi_{k-1} + \Psi_{k-1,k} + \Psi_{k-1,k}^{(0,k-1)}, \tag{3.19}
$$

where $\Psi_0 = 0$, $\Psi_{0,1}^{(0,0)} = 0$ and

$$\Psi_{k-1,k} := \int_{t_{k-1}}^{t_k} A_t \left(\int_{t_{k-1}}^{t} B_s(\rho dW_{1,s} + \sqrt{1-\rho^2} dW_{2,s}) \right) dW_{1t}$$
$$- \int_{t_{k-1}}^{t_k} C_t \left(\int_{t_{k-1}}^{t} B_s(\rho dW_{1,s} + \sqrt{1-\rho^2} dW_{2,s}) \right) dt,$$
$$\Psi_{k-1,k}^{(0,k-1)} := \left(\int_0^{t_{k-1}} B_s(\rho dW_{1,s} + \sqrt{1-\rho^2} dW_{2,s}) \right) \left(\int_{t_{k-1}}^{t_k} A_t dW_{1t} \right)$$
$$- \left(\int_0^{t_{k-1}} B_s(\rho dW_{1,s} + \sqrt{1-\rho^2} dW_{2,s}) \right) \left(\int_{t_{k-1}}^{t_k} C_t dt \right) \equiv \sum_{l=1}^{k-1} \Psi_{k-1,k}^{(l-1,l)},$$
$$\Psi_{k-1,k}^{(l-1,l)} := \left(\int_{t_{l-1}}^{t_l} B_s(\rho dW_{1,s} + \sqrt{1-\rho^2} dW_{2,s}) \right) \left(\int_{t_{k-1}}^{t_k} A_t dW_{1t} \right)$$
$$- \left(\int_{t_{l-1}}^{t_l} B_s(\rho dW_{1,s} + \sqrt{1-\rho^2} dW_{2,s}) \right) \left(\int_{t_{k-1}}^{t_k} C_t dt \right),$$

$1 \le l \le k-1$.

Applying the integration by parts formula, we have

$$\pi_{i,j} = D_1^* \left(\Psi_i \gamma_{ij}^{X^{(0)}} D_1 \int_0^{t_j} V(\sigma_t^{(0)}, t) dW_{1,t} \right)$$
$$= \gamma_{ij}^{X^{(0)}} \left[\Psi_i \int_0^T V(\sigma_u^{(0)}, u) \mathbf{1}_{u \le t_j} dW_{1,u} - \int_0^T D_{u,1} \Psi_i \mathbf{1}_{u \le t_i} V(\sigma_u^{(0)}, u) \mathbf{1}_{t \le t_j} du \right]. \tag{3.20}$$

The Malliavin weight for $Barrier_N^{SV}$ is given by

$$\pi = \sum_{i,j=1}^{N} \pi_{i,j} = \pi_{1,1} + \sum_{k=2}^{N} (\pi_{k,k} + \pi_{k,k+1} + \pi_{k+1,k}). \tag{3.21}$$

For $i = 1, \dots, N-1$,

$$\pi_{i,i} = \left(\frac{1}{\Sigma_{(i-1,i)}} + \frac{1}{\Sigma_{(i,i+1)}}\right)\left[\Psi_i \int_0^{t_i} V(\sigma_u^{(0)}, u)dW_{1,u} - \int_0^{t_i} D_{u,1}\Psi_i V(\sigma_u^{(0)}, u)du\right],$$

$$\pi_{i,i+1} = \left(-\frac{1}{\Sigma_{(i,i+1)}}\right)\left[\Psi_i \int_0^{t_{i+1}} V(\sigma_u^{(0)}, u)dW_{1,u} - \int_0^{t_i} D_{u,1}\Psi_i V(\sigma_u^{(0)}, u)du\right],$$

$$\pi_{i+1,i} = \left(-\frac{1}{\Sigma_{(i,i+1)}}\right)\left[\Psi_{i+1} \int_0^{t_i} V(\sigma_u^{(0)}, u)dW_{1,u} - \int_0^{t_i} D_{u,1}\Psi_{i+1} V(\sigma_u^{(0)}, u)du\right], \tag{3.22}$$

and for N,

$$\pi_{N,N} = \frac{1}{\Sigma_{(N-1,N)}}\left[\Psi_N \int_0^{t_N} V(\sigma_u^{(0)}, u)dW_{1,u} - \int_0^{t_N} D_{u,1}\Psi_N V(\sigma_u^{(0)}, u)du\right]. \tag{3.23}$$

Thus, for $i = 2, \ldots, N-1$,

$$\begin{aligned}
&\pi_{i-1,i} + \pi_{i,i-1} + \pi_{i,i} \\
&= \left(-\frac{1}{\Sigma_{(i-1,i)}}\right)\left[\Psi_{i-1} \int_0^{t_{i-1}} V(\sigma_u^{(0)}, u)dW_{1,u} - \int_0^{t_{i-1}} D_{u,1}\Psi_i V(\sigma_u^{(0)}, u)du\right] \\
&\quad + \left(\frac{1}{\Sigma_{(i-1,i)}}\right)\left[\left(\Psi_{i-1,i} + \Psi_{i-1,i}^{(0,i-1)}\right) \int_{t_{i-1}}^{t_i} V(\sigma_u^{(0)}, u)dW_{1,u}\right. \\
&\quad \left. - \int_{t_{i-1}}^{t_i} D_{u,1}\left(\Psi_{i-1,i} + \Psi_{i-1,i}^{(0,i-1)}\right) V(\sigma_u^{(0)}, u)du\right] \\
&\quad + \left(\frac{1}{\Sigma_{(i,i+1)}}\right)\left[\Psi_i \int_0^{t_i} V(\sigma_u^{(0)}, u)dW_{1,u} - \int_0^{t_i} D_{u,1}\Psi_{i+1} V(\sigma_u^{(0)}, u)du\right],
\end{aligned} \tag{3.24}$$

and for N,

$$\begin{aligned}
&\pi_{N-1,N} + \pi_{N,N-1} + \pi_{N,N} \\
&= \left(-\frac{1}{\Sigma_{(N-1,N)}}\right)\left[\Psi_{N-1} \int_0^{t_{N-1}} V(\sigma_u^{(0)}, u)dW_{1,u} - \int_0^{t_{N-1}} D_{u,1}\Psi_i V(\sigma_u^{(0)}, u)du\right]
\end{aligned}$$

$$+\left(\frac{1}{\Sigma_{(N-1,N)}}\right)\left[\left(\Psi_{N-1,N}+\Psi_{N-1,N}^{(0,N-1)}\right)\int_{t_{N-1}}^{t_N} V(\sigma_u^{(0)},u)dW_{1,u}\right.$$
$$\left.-\int_{t_{N-1}}^{t_N} D_{u,1}\left(\Psi_{N-1,N}+\Psi_{N-1,N}^{(0,N-1)}\right)V(\sigma_u^{(0)},u)du\right]. \tag{3.25}$$

Hence, we have

$$\pi=\sum_{k=1}^{N}\left(\frac{1}{\Sigma_{k-1,k}}\right)\left\{\left(\Psi_{k-1,k}+\Psi_{k-1,k}^{(0,k-1)}\right)\int_{t_{k-1}}^{t_k} V(\sigma_u^{(0)},u)dW_{1,u}\right.$$
$$\left.-\int_{t_{k-1}}^{t_k} D_{u,1}\left(\Psi_{k-1,k}+\Psi_{k-1,k}^{(0,k-1)}\right)V(\sigma_u^{(0)},u)du\right\}$$
$$=\sum_{k=1}^{N}\left(\frac{1}{\Sigma_{k-1,k}}\right)\left\{\Psi_{k-1,k}\int_{t_{k-1}}^{t_k} V(\sigma_u^{(0)},u)dW_{1,u}-\int_{t_{k-1}}^{t_k} D_{u,1}\Psi_{k-1,k}V(\sigma_u^{(0)},u)du\right\}$$
$$+\sum_{k=2}^{N}\sum_{l=1}^{k-1}\left(\frac{1}{\Sigma_{k-1,k}}\right)\left\{\Psi_{k-1,k}^{(l-1,l)}\int_{t_{k-1}}^{t_k} V(\sigma_u^{(0)},u)dW_{1,u}\right.$$
$$\left.-\int_{t_{k-1}}^{t_k} D_{u,1}\Psi_{k-1,k}^{(l-1,l)}V(\sigma_u^{(0)},u)du\right\}. \tag{3.26}$$

In order to obtain a closed form approximation of $Barrier_N^{SV}$, we calculate the push down of (3.26) in the following manner. For $t_{k-1}<u\le t_k$, the Malliavin derivatives of $\Psi_{k-1,k}$ and $\Psi_{k-1,k}^{(l-1,l)}$ are calculated as follows;

$$D_{u,1}\Psi_{k-1,k}=D_{u,1}\int_{t_{k-1}}^{t_k} A_t\int_{t_{k-1}}^{t} B_s(\rho dW_{1,s}+\sqrt{1-\rho^2}dW_{2,s})dW_{1,t}$$
$$-D_{u,1}\int_{t_{k-1}}^{t_k} C_t\int_{t_{k-1}}^{t} B_s(\rho dW_{1,s}+\sqrt{1-\rho^2}dW_{2,s})dt$$
$$=\rho A_u\int_{t_{k-1}}^{u} B_s dW_{1,s}+\rho B_u\int_{u}^{t_k} A_t dW_{1,t}, \tag{3.27}$$

$$D_{u,1}\Psi_{k-1,k}^{(l-1,l)}=D_{u,1}\left(\int_{t_{l-1}}^{t_l} B_s(\rho dW_{1,s}+\sqrt{1-\rho^2}dW_{2,s})\right)\left(\int_{t_{k-1}}^{t_k} A_t dW_{1,t}\right)$$

$$-D_{u,1}\left(\int_{t_{l-1}}^{t_l} B_s(\rho dW_{1,s}+\sqrt{1-\rho^2}dW_{2,s})\right)\left(\int_{t_{k-1}}^{t_k} C_t dt\right)$$

$$= A_u\left(\int_{t_{l-1}}^{t_l} B_s(\rho dW_{1,s}+\sqrt{1-\rho^2}dW_{2,s})\right). \quad (3.28)$$

Then, we have

$$\int_{t_{k-1}}^{t_k} D_{u,1}\Psi_{k-1,k}V(\sigma_u^{(0)},u)du = \rho\int_{t_{k-1}}^{t_k} A_u V(\sigma_u^{(0)},u)\int_{t_{k-1}}^{u} B_s dW_{1,s}du$$
$$+\rho\int_{t_{k-1}}^{t_k} B_u V(\sigma_u^{(0)},u)\int_{u}^{t_k} A_t dW_{1,t}du$$
$$-\rho\int_{t_{k-1}}^{t_k} B_u V(\sigma_u^{(0)},u)\int_{u}^{t_k} C_t dtdu,$$

$$\int_{t_{k-1}}^{t_k} D_{u,1}\Psi_{k-1,k}^{(l-1,l)}V(\sigma_u^{(0)},u)du = \left(\int_{t_{l-1}}^{t_l} B_s(\rho dW_{1,s}+\sqrt{1-\rho^2}dW_{2,s})\right)$$
$$\left(\int_{t_{k-1}}^{t_k} A_u V(\sigma_u^{(0)},u)du\right).$$

Taking the conditional expectations, we obtain the followings:

$$E\left[\Psi_{k-1,k}\int_{t_{k-1}}^{t_k} V(\sigma_u^{(0)},u)dW_{1,u}|Y=y\right] = \zeta_{k-1,k}\left(\frac{(y_k-y_{k-1})^2}{\Sigma_{k-1,k}^2}-\frac{(y_k-y_{k-1})}{\Sigma_{k-1,k}}\right)$$
$$\times(y_k-y_{k-1}),$$

$$E\left[\int_{t_{k-1}}^{t_k} D_{u,1}\Psi_{k-1,k}V(\sigma_u^{(0)},u)du|Y=y\right] = \zeta_{k-1,k}\left(\frac{2(y_k-y_{k-1})}{\Sigma_{k-1,k}}-1\right),$$

$$E\left[\Psi_{k-1,k}^{(l-1,l)}\int_{t_{k-1}}^{t_k} V(\sigma_u^{(0)},u)dW_{1,u}|Y=y\right] = \zeta_{k-1,k}^{(l-1,l)}\left(\frac{(y_l-y_{l-1})}{\Sigma_{l-1,l}}\right)\left(\frac{(y_k-y_{k-1})}{\Sigma_{k-1,k}}-1\right)$$
$$\times(y_k-y_{k-1}),$$

$$E\left[\int_{t_{k-1}}^{t_k} D_{u,1}\Psi_{k-1,k}^{(l-1,l)}V(\sigma_u^{(0)},u)du|Y=y\right] = \zeta_{k-1,k}^{(l-1,l)}\left(\frac{(y_l-y_{l-1})}{\Sigma_{l-1,l}}\right)(y_k-y_{k-1}).$$

Here, we use the following formulas: (e.g. see Section 3 in Takahashi et al. (2009) that gives more general results and their derivation.)

$$E\left[\int_0^T q_{2t}^{'} dW_t | \int_0^T q_{1v}^{'} dW_v = x\right] = \left(\int_0^T q_{2t}^{'} q_{1t} dt\right) \frac{x}{\Sigma}, \tag{3.29}$$

$$E\left[\int_0^T \int_0^t q_{2u}^{'} dW_u q_{3t}^{'} dW_t | \int_0^T q_{1v}^{'} dW_v = x\right] = \left(\int_0^T \int_0^t q_{2u}^{'} q_{1u} du q_{3t}^{'} q_{1t} dt\right) \frac{x^2 - \Sigma}{\Sigma^2}, \tag{3.30}$$

where $q_i \in L^2[0, T]$, $i = 1, 2, 3$ and $\Sigma = \int_0^T |q_{1t}|^2 dt > 0$.

Therefore, we obtain

$$\begin{aligned}\vartheta(y_1, \ldots, y_N) &= E[\pi | Y = y] \\ &= \sum_{k=1}^{N} \zeta_{k-1,k} \left(\frac{(y_k - y_{k-1})^3}{\Sigma_{k-1,k}^3} - \frac{3(y_k - y_{k-1})}{\Sigma_{k-1,k}^2} - \right. \\ &\quad \left. \frac{(y_k - y_{k-1})^2}{\Sigma_{k-1,k}^2} + \frac{1}{\Sigma_{k-1,k}} \right) \\ &\quad + \sum_{k=2}^{N} \sum_{l=1}^{k-1} \zeta_{k-1,k}^{(l-1,l)} \left(\frac{y_l - y_{l-1}}{\Sigma_{l-1,l}} \right) \\ &\quad \left(\frac{(y_k - y_{k-1})^2}{\Sigma_{k-1,k}^2} - \frac{1}{\Sigma_{k-1,k}} - \frac{y_k - y_{k-1}}{\Sigma_{k-1,k}} \right).\end{aligned} \tag{3.31}$$

□

3.3 Application to CEV Model with General Stochastic Volatility

Next, consider CEV Model with General Stochastic Volatility.

$$\begin{aligned} dS_t^{(\epsilon)} &= \alpha S_t^{(\epsilon)} dt + V(\sigma_t^{(\epsilon)}, t) \left(S_t^{(\epsilon)}\right)^{\beta} dW_{1,t}, \\ d\sigma_t^{(\epsilon)} &= A_0(\sigma_t^{(\epsilon)}, t) dt + \epsilon A_1(\sigma_t^{(\epsilon)}, t)(\rho dW_{1,t} + \sqrt{1-\rho^2} dW_{2,t}). \end{aligned}$$

In this case, we cannot apply the asymptotic expansion directly, because the density at $\epsilon = 0$ is not the Gaussian. However, if we introduce a perturbation for the asset dynamics,

$$\begin{aligned} dS_t^{(\epsilon)} &= \alpha S_t^{(\epsilon)} dt + \epsilon V(\sigma_t^{(\epsilon)}, t) \left(S_t^{(\epsilon)}\right)^{\beta} dW_{1,t}, \\ d\sigma_t^{(\epsilon)} &= A_0(\sigma_t^{(\epsilon)}, t) dt + \epsilon A_1(\sigma_t^{(\epsilon)}, t)(\rho dW_{1,t} + \sqrt{1-\rho^2} dW_{2,t}), \end{aligned}$$

we can expand the density for the transformed process $Z_{t_i} := \frac{1}{\epsilon}\left(S_{t_i}^{(\epsilon)} - S_{t_i}^{(0)}\right)$ as follows,

$$
\begin{aligned}
& p^{Z^{(\epsilon)}}(z_1, \ldots, z_N) \\
&= E\left[\delta_z\left(Z_{t_1}^{(\epsilon)}, \ldots, Z_{t_N}^{(\epsilon)}\right)\right] \\
&= E\left[\delta_z\left(Z_{t_1}^{(0)}, \ldots, Z_{t_N}^{(0)}\right)\right] \\
&\quad + \sum_{j=1}^{m} \epsilon^j \sum_{k}^{(j)} E\left[\delta_z\left(Z_{t_1}^{(0)}, \ldots, Z_{t_N}^{(0)}\right) H_{\alpha^{(k)}}\left(Z^{(0)}, \prod_{l=1}^{k} Z_{\beta_l}^{0,\alpha_l}\right)\right] + O(\epsilon^{m+1}) \\
&= p^{Z^{(0)}}(z_1, \ldots, z_N) \\
&\quad + \sum_{j=1}^{m} \epsilon^j \sum_{k}^{(j)} E\left[H_{\alpha^{(k)}}\left(Z^{(0)}, \prod_{l=1}^{k} Z_{\beta_l}^{0,\alpha_l}\right) | Z^{(0)} = (z_1, \ldots, z_N)\right] p^{Z^{(0)}} \\
&\quad (z_1, \ldots, z_N) + O(\epsilon^{m+1}),
\end{aligned} \tag{3.32}
$$

where $Z_k^{0,i} := \frac{1}{k!}\frac{d^k}{d\epsilon^k} Z_{t_i}^{(\epsilon)}|_{\epsilon=0}, k \in \mathbf{N}, i = 1, \ldots, N$. Note that the Malliavin covariance matrix $\sigma = [\sigma_{(i,j)}]_{1 \le i,j \le N}$ of $Z^{(0)}$ is given by

$$
\begin{aligned}
\sigma_{(i,j)} &= \left\langle D\left(\int_0^T e^{\alpha t_i} e^{-\alpha t} V(\sigma_t^{(0)}, t)(S_t^{(0)})^{\beta} 1_{\{t \le t_i\}} dW_{1t}\right),\right. \\
&\qquad \left. D\left(\int_0^T e^{\alpha t_j} e^{-\alpha t} V(\sigma_t^{(0)}, t)(S_t^{(0)})^{\beta} 1_{\{t \le t_j\}} dW_{1t}\right)\right\rangle_H \\
&= \int_0^T \left(e^{\alpha t_i} e^{-\alpha s} V(\sigma_s^{(0)}, s)(S_s^{(0)})^{\beta} 1_{\{s \le t_i\}}\right)\left(e^{\alpha t_j} e^{-\alpha s} V(\sigma_s^{(0)}, s)(S_s^{(0)})^{\beta} 1_{\{s \le t_j\}}\right) ds \\
&= e^{\alpha(t_i + t_j)} \int_0^{t_i \wedge t_j} e^{-2\alpha s} V(\sigma_s^{(0)}, s)^2 (S_s^{(0)})^{2\beta} ds, \ (1 \le i, j \le N).
\end{aligned} \tag{3.33}
$$

Then the N-dimensional normal density function $p^{Z^0}(z_1, \ldots, z_N)$ of $Z^{(0)} = (Z_{t_1}^{(0)}, \ldots, Z_{t_N}^{(0)})$ is given by

$$
\begin{aligned}
p^{Z^{(0)}}(z_1, \ldots, z_N) &= \frac{1}{(2\pi)^{N/2}\left(e^{\alpha \sum_{i=1}^{N} t_i}\right)\Sigma_{(0,1)}^{1/2} \cdots \Sigma_{(N-1,N)}^{1/2}} \\
&\qquad \exp\left(-\sum_{i=1}^{N} \frac{(e^{-\alpha t_i} z_i - e^{-\alpha t_{i-1}} z_{i-1})^2}{2\Sigma_{i-1,i}}\right)
\end{aligned} \tag{3.34}
$$

with $t_0 = z_0 = 0$ and

$$\Sigma_{(i-1,i)} = \int_{t_{i-1}}^{t_i} e^{-2\alpha s} V(\sigma_s^{(0)}, s)^2 (S_s^{(0)})^{2\beta} ds, \ (1 \le i \le N). \tag{3.35}$$

Therefore, we can evaluate the expectation analytically for CEV model with general stochastic volatility as follows:

$$Barrier_N^{SABR} = e^{-rT} \int_{\mathbf{R}^N} \psi(z_1, \ldots, z_N) p^{Z^{(\epsilon)}}(z_1, \ldots, z_N) dz_1 \ldots dz_N, \tag{3.36}$$

where

$$\psi(y_1, \ldots, y_N) = \mathbf{1}_{\left\{S_{t_1}^{(0)} + \epsilon z_1 \in B\right\}} \cdots \mathbf{1}_{\left\{S_{t_N}^{(0)} + \epsilon z_N \in B\right\}} \epsilon (K^\epsilon + z_N)^+, \tag{3.37}$$

with

$$K^\epsilon = \frac{1}{\epsilon} \left(S_{t_N}^{(0)} - K \right). \tag{3.38}$$

4 Numerical Examples

This section provides numerical examples for pricing double barrier call options with discrete monitoring under Heston model and λ-SABR model.

4.1 Heston Model

First, we deal with Heston model. The form of Heston model is as follows.

$$dS_t^{(\epsilon)} = r S_t + \sqrt{\sigma_t^{(\epsilon)}} S_t^{(\epsilon)} dW_{1,t}, \tag{4.1}$$

$$d\sigma_t^{(\epsilon)} = \kappa(\theta - \sigma_t^{(\epsilon)}) dt + \epsilon \sqrt{\sigma_t^{(\epsilon)}} (\rho dW_{1,t} + \sqrt{1-\rho^2} dW_{2,t}). \tag{4.2}$$

The time to maturity of the options is $T = 1$, and the monitoring dates when one judges whether the underlying price hits the barriers is set to be 0.25, 0.5, 0.75, 1.0 (case A) and 0.5, 1.0 (case B). The parameters are set to be the following: The initial underlying asset's price and variance are $S_0^{(\epsilon)} = 100$ and $\sigma_0^{(\epsilon)} = 0.02$, respectively. The mean reversion speed and level are set to be $\kappa = 1$ and $\theta = 0.02$ respectively. The lower barrier L and the upper barrier U are set to be $L = 80$ and $U = 120$ respectively. Also, the riskless interest rate (r) is set to be 0.

The volatility on the variance ϵ, the correlation between the underlying asset and the variance ρ and the strike price vary for the following cases:

I: $\epsilon = 0.02$, II: $\epsilon = 0.05$, III: $\epsilon = 0.10$, IV: $\epsilon = 0.15$, V: $\epsilon = 0.20$,
i: $\rho = -0.7$, ii: $\rho = 0.0$, iii: $\rho = 0.7$,
1: $K = 90$, 2: $K = 100$, 3: $K = 110$.

For each double barrier option the range of the integration is bounded from above and below. Thus, the Gauss-Legendre Quadrature is used for efficient computation.

For the single barrier case (that the integration range includes infinity), it is more efficient to use the Gauss - Laguerre Quadrature with Gauss - Legendre Quadrature than to use only the Gauss - Legendre Quadrature.

Note that Gauss - Legendre Quadrature is given by

$$\int_a^b f(x)dx = \frac{a-b}{2}\int_{-1}^{1} f\left(\frac{a-b}{2}z + \frac{a+b}{2}\right)dz \sim w(z_k) f\left(\frac{a-b}{2}z_k + \frac{a+b}{2}\right),$$

where $z_k, k = 1, \ldots, n$ are the values such that $P_n(z_k) = 0$ and P_n denotes the Legendre polynomial of the n-th order. The weights, $w(z_k)$ are obtained by

$$w(z_k) = 2/[nP_{n-1}(z_k)P_n^{'}(z_k)].$$

For our computation, set $n = 20$. Gauss - Legendre Quadrature can calculate the integration with smaller number of computation than other usual computational methods(e.g. trapezoidal rule). Thus, the speed of calculation is very fast.

For example in the case A, at $t = 0.75$ for each $y_{N-1} = z_k, k = 1, \ldots, 20$, the payoff at maturity $T = 1$;

$$se^{y_N - \frac{1}{2}\xi_N} - K$$

is integrated from $\max\{L, K\}$ to U.

The values obtained by the previous integration for $z_k, k = 1, \ldots, 20$ at $t = 0.75$ is used for the integration from L to U at $t = 0.50$ for each z_k. Recursively in this way, the value for the initial value at $t = 0$ is obtained.

The results are shown in Table 1 (case A) and Table 2 (case B) below. MC denotes the benchmark price computed by Monte Carlo simulation. Except for the cases that $\rho = 0$, our first order expansions improve the accuracies relative to the Black-Scholes model(BS) where the stochastic volatility component is ignored: Note that when $\rho = 0$, the approximations by the first order expansion are equivalent to those by the Black-Scholes model(BS) and that the Black-Scholes model provides relatively good approximations when $\rho = 0$.

4.2 λ-SABR Model

The next numerical example is based on λ-SABR model which is described as follows.

$$dS_t^{(\epsilon)} = rS_t + \sigma_t^{(\epsilon)} S_t^{\beta(\epsilon)} dW_{1,t}, \tag{4.3}$$

$$d\sigma_t^{(\epsilon)} = \lambda(\theta - \sigma_t^{(\epsilon)})dt + \epsilon\sigma_t^{(\epsilon)}(\rho dW_{1,t} + \sqrt{1-\rho^2}dW_{2,t}). \tag{4.4}$$

Table 1 Discrete barrier option prices (case A)

	AE	BS	MC	AE error	BS error	AE error rate	BS error rate
I-i-1	8.06	7.86	8.06	−0.01	−0.21	−0.1	−2.6
I-i-2	2.96	2.85	2.96	−0.00	−0.12	−0.1	−3.9
I-i-3	0.54	0.52	0.54	−0.00	−0.03	−0.1	−4.7
I-ii-1	7.86	7.86	7.86	−0.00	−0.00	0.0	0.0
I-ii-2	2.85	2.85	2.85	−0.00	−0.00	0.0	0.0
I-ii-3	0.52	0.52	0.52	0.00	0.00	0.1	0.1
I-iii-1	7.65	7.86	7.66	−0.01	0.19	−0.1	2.5
I-iii-2	2.74	2.85	2.74	−0.00	0.11	−0.2	3.9
I-iii-3	0.49	0.52	0.49	−0.00	0.02	−0.2	4.8
II-i-1	8.36	7.86	8.40	−0.04	−0.55	−0.5	−6.5
II-i-2	3.13	2.85	3.15	−0.02	−0.30	−0.7	−9.6
II-i-3	0.58	0.52	0.58	−0.00	−0.07	−0.8	−11.4
II-ii-1	7.86	7.86	7.87	−0.02	−0.02	−0.2	−0.2
II-ii-2	2.85	2.85	2.85	0.00	0.00	0.0	0.0
II-ii-3	0.52	0.52	0.51	0.00	0.00	0.5	0.5
II-iii-1	7.35	7.86	7.39	−0.04	0.46	−0.6	6.3
II-iii-2	2.57	2.85	2.59	−0.02	0.26	−0.8	10.0
II-iii-3	0.45	0.52	0.46	−0.00	0.06	−1.0	12.4
III-i-1	8.86	7.86	9.05	−0.18	−1.19	−2.0	−13.1
III-i-2	3.40	2.85	3.49	−0.09	−0.65	−2.5	−18.5
III-i-3	0.64	0.52	0.66	−0.02	−0.14	−2.6	−21.3
III-ii-1	7.86	7.86	7.92	−0.06	−0.06	−0.8	−0.8
III-ii-2	2.85	2.85	2.84	0.01	0.01	0.3	0.3
III-ii-3	0.52	0.52	0.50	0.01	0.01	2.5	2.5
III-iii-1	6.85	7.86	7.00	−0.15	0.86	−2.1	12.3
III-iii-2	2.29	2.85	2.36	−0.07	0.49	−3.0	20.7
III-iii-3	0.39	0.52	0.41	−0.02	0.11	−3.9	26.2
IV-i-1	9.37	7.86	9.74	−0.37	−1.88	−3.8	−19.3
IV-i-2	3.68	2.85	3.85	−0.16	−1.00	−4.3	−26.0
IV-i-3	0.70	0.52	0.72	−0.02	−0.20	−2.6	−28.3
IV-ii-1	7.86	7.86	7.98	−0.13	−0.13	−1.6	−1.6
IV-ii-2	2.85	2.85	2.82	0.03	0.03	1.0	1.0
IV-ii-3	0.52	0.52	0.49	0.03	0.03	6.1	6.1
IV-iii-1	6.34	7.86	6.66	−0.32	1.19	−4.8	17.9
IV-iii-2	2.01	2.85	2.15	−0.14	0.70	−6.5	32.4
IV-iii-3	0.33	0.52	0.36	−0.03	0.15	−8.9	41.9
V-i-1	9.87	7.86	10.34	−0.46	−2.48	−4.5	−24.0
V-i-2	3.96	2.85	4.12	−0.15	−1.27	−3.8	−30.9
V-i-3	0.76	0.52	0.74	0.03	−0.22	3.6	−29.9

Table 1 continued

	AE	BS	MC	AE error	BS error	AE error rate	BS error rate
V-ii-1	7.86	7.86	8.05	−0.20	−0.20	−2.5	−2.5
V-ii-2	2.85	2.85	2.78	0.07	0.07	2.3	2.3
V-ii-3	0.52	0.52	0.46	0.05	0.05	11.5	11.5
V-iii-1	5.84	7.86	6.39	−0.55	1.46	−8.7	22.9
V-iii-2	1.73	2.85	1.96	−0.23	0.89	−11.6	45.4
V-iii-3	0.27	0.52	0.32	−0.05	0.19	−16.2	60.4

Table 2 Discrete barrier option prices (case B)

	AE	BS	MC	AE error	BS error	AE error rate	BS error rate
I-i-1	8.38	8.19	8.39	−0.01	−0.20	−0.1	−2.4
I-i-2	3.15	3.04	3.15	−0.01	−0.11	−0.2	−3.6
I-i-3	0.60	0.58	0.61	−0.00	−0.03	−0.2	−4.2
I-ii-1	8.19	8.19	8.19	−0.00	−0.00	−0.1	−0.1
I-ii-2	3.04	3.04	3.04	−0.00	−0.00	−0.1	−0.1
I-ii-3	0.58	0.58	0.58	0.00	0.00	0.0	0.0
I-iii-1	8.00	8.19	8.01	−0.01	0.18	−0.1	2.3
I-iii-2	2.93	3.04	2.94	−0.01	0.10	−0.2	3.5
I-iii-3	0.56	0.58	0.56	−0.00	0.02	−0.2	4.1
II-i-1	8.67	8.19	8.71	−0.04	−0.52	−0.5	−6.0
II-i-2	3.31	3.04	3.33	−0.02	−0.29	−0.6	−8.6
II-i-3	0.64	0.58	0.64	−0.00	−0.06	−0.6	−9.9
II-ii-1	8.19	8.19	8.21	−0.02	−0.02	−0.2	−0.2
II-ii-2	3.04	3.04	3.04	0.00	0.00	0.0	0.0
II-ii-3	0.58	0.58	0.58	0.00	0.00	0.5	0.5
II-iii-1	7.71	8.19	7.75	−0.04	0.44	−0.5	5.7
II-iii-2	2.77	3.04	2.79	−0.02	0.25	−0.6	8.9
II-iii-3	0.52	0.58	0.52	−0.00	0.06	−0.8	10.7
III-i-1	9.14	8.19	9.30	−0.16	−1.11	−1.7	−12.0
III-i-2	3.57	3.04	3.64	−0.07	−0.60	−2.0	−16.6
III-i-3	0.70	0.58	0.71	−0.01	−0.13	−1.4	−18.3
III-ii-1	8.19	8.19	8.24	−0.05	−0.05	−0.6	−0.6
III-ii-2	3.04	3.04	3.02	0.02	0.02	0.6	0.6
III-ii-3	0.58	0.58	0.56	0.02	0.02	2.8	2.8
III-iii-1	7.23	8.19	7.36	−0.13	0.83	−1.8	11.2
III-iii-2	2.51	3.04	2.56	−0.06	0.48	−2.2	18.6
III-iii-3	0.46	0.58	0.47	−0.01	0.11	−2.6	22.7
IV-i-1	9.62	8.19	9.94	−0.31	−1.75	−3.2	−17.6
IV-i-2	3.84	3.04	3.96	−0.12	−0.92	−3.1	−23.3
IV-i-3	0.76	0.58	0.76	−0.00	−0.18	−0.2	−23.8

Table 2 continued

	AE	BS	MC	AE error	BS error	AE error rate	BS error rate
IV-ii-1	8.19	8.19	8.29	−0.10	−0.10	−1.2	−1.2
IV-ii-2	3.04	3.04	2.99	0.05	0.05	1.6	1.6
IV-ii-3	0.58	0.58	0.54	0.04	0.04	6.6	6.6
IV-iii-1	6.76	8.19	7.04	−0.28	1.15	−4.0	16.4
IV-iii-2	2.24	3.04	2.36	−0.11	0.68	−4.9	29.1
IV-iii-3	0.40	0.58	0.43	−0.03	0.15	−5.9	36.3
V-i-1	10.10	8.19	10.48	−0.38	−2.29	−3.6	−21.9
V-i-2	4.11	3.04	4.20	−0.09	−1.16	−2.2	−27.6
V-i-3	0.82	0.58	0.76	0.06	−0.18	7.2	−24.1
V-ii-1	8.19	8.19	8.34	−0.15	−0.15	−1.9	−1.9
V-ii-2	3.04	3.04	2.94	0.10	0.10	3.3	3.3
V-ii-3	0.58	0.58	0.52	0.06	0.06	12.5	12.5
V-iii-1	6.28	8.19	6.77	−0.49	1.42	−7.2	21.0
V-iii-2	1.97	3.04	2.16	−0.19	0.88	−8.6	40.7
V-iii-3	0.34	0.58	0.38	−0.04	0.20	−10.7	52.2

We consider the cases of $\beta = 1$ (case C) and $\beta = 0.5$ (case D). The time to maturity of the options are $T = 1$ and the monitoring dates are 0.25, 0.5, 0.75, 1.0. The parameters are set to be the following: The initial underlying asset's price is $S_0^{(\epsilon)} = 100$ for both cases. The initial volatilities are $\sigma_0^{(\epsilon)} = 0.15$ for (case C) and $\sigma_0^{(\epsilon)} = 1.5$ for (case D), respectively. The mean reversion speed and level are set to be $\lambda = 1, \theta = 0.15$ for (case C) and $\theta = 1.5$ for (case D), respectively. The lower and upper barriers are set to be $L = 80$ and $U = 120$, respectively. The riskless interest rate (r) is set to be 0.

The volatility on the volatility ϵ, the correlation ρ between the underlying asset price and the volatility as well as the strike prices are set to be the same as in Heston model.

The results are shown in Table 3 (case C) and Table 4 (case D) below. MC denotes the benchmark price computed by Monte Carlo simulation.

The results show that the second order expansions provide better approximations than the first order expansions. Also, we note that the approximations for $\beta = 0.5$(case D) are better than those for $\beta = 1$(case C). It is because the λ-SABR model is expanded around a normal distribution and the distribution of the underlying asset price is closer to a normal when β is closer to zero. Thus, the smaller β gives better approximation in general.

It is generally observed in both Heston and λ-SABR models, the higher volatility on the variance (or volatility) cause worse approximation, especially when the correlation is not 0 and the strike price is in-the-money($K = 90$). It implies that the higher order expansion may be necessary for those cases.

Table 3 Discrete barrier option prices (case C)

	AE1	AE2	MC	AE1 error	AE2 error	AE1 error rate	AE2 error rate
I-i-1	8.30	7.49	7.54	0.75	−0.06	10.0	−0.8
I-i-2	3.19	2.73	2.77	0.42	−0.04	15.2	−1.3
I-i-3	0.61	0.50	0.51	0.09	−0.01	18.5	−2.1
I-ii-1	8.30	7.42	7.49	0.81	−0.06	10.8	−0.8
I-ii-2	3.19	2.70	2.74	0.45	−0.04	16.5	−1.5
I-ii-3	0.61	0.49	0.51	0.10	−0.01	20.2	−2.3
I-iii-1	8.30	7.36	7.43	0.86	−0.07	11.6	−0.9
I-iii-2	3.19	2.66	2.71	0.48	−0.04	17.8	−1.7
I-iii-3	0.61	0.49	0.50	0.11	−0.01	21.8	−2.5
II-i-1	8.30	7.58	7.63	0.67	−0.05	8.7	−0.7
II-i-2	3.19	2.78	2.81	0.37	−0.03	13.3	−1.1
II-i-3	0.61	0.51	0.52	0.08	−0.01	16.1	−1.7
II-ii-1	8.30	7.42	7.49	0.81	−0.06	10.8	−0.9
II-ii-2	3.19	2.70	2.74	0.45	−0.04	16.5	−1.5
II-ii-3	0.61	0.49	0.51	0.10	−0.01	20.1	−2.3
II-iii-1	8.30	7.27	7.35	0.94	−0.08	12.8	−1.1
II-iii-2	3.19	2.61	2.66	0.53	−0.05	19.7	−2.0
II-iii-3	0.61	0.47	0.49	0.12	−0.02	24.2	−3.1
III-i-1	8.30	7.73	7.77	0.52	−0.04	6.7	−0.6
III-i-2	3.19	2.87	2.89	0.29	−0.03	10.2	−0.9
III-i-3	0.61	0.53	0.54	0.07	−0.01	12.3	−1.2
III-ii-1	8.30	7.42	7.49	0.81	−0.06	10.8	−0.8
III-ii-2	3.19	2.70	2.74	0.45	−0.04	16.5	−1.4
III-ii-3	0.61	0.49	0.50	0.10	−0.01	20.3	−2.2
III-iii-1	8.30	7.12	7.22	1.07	−0.10	14.9	−1.4
III-iii-2	3.19	2.52	2.59	0.60	−0.07	23.1	−2.5
III-iii-3	0.61	0.45	0.47	0.13	−0.02	28.5	−4.0
IV-i-1	8.30	7.88	7.92	0.38	−0.04	4.8	−0.5
IV-i-2	3.19	2.95	2.97	0.21	−0.02	7.2	−0.6
IV-i-3	0.61	0.55	0.56	0.05	−0.00	8.7	−0.8
IV-ii-1	8.30	7.42	7.49	0.81	−0.06	10.8	−0.8
IV-ii-2	3.19	2.70	2.73	0.45	−0.04	16.6	−1.4
IV-ii-3	0.61	0.49	0.50	0.10	−0.01	20.5	−2.0
IV-iii-1	8.30	6.97	7.09	1.20	−0.12	17.0	−1.7
IV-iii-2	3.19	2.44	2.52	0.67	−0.08	26.6	−3.2
IV-iii-3	0.61	0.43	0.46	0.15	−0.02	33.1	−5.0
V-i-1	8.30	8.03	8.06	0.23	−0.03	2.9	−0.4
V-i-2	3.19	3.04	3.06	0.13	−0.01	4.4	−0.5
V-i-3	0.61	0.57	0.58	0.03	−0.00	5.2	−0.5
V-ii-1	8.30	7.42	7.48	0.81	−0.06	10.8	−0.8

Table 3 continued

	AE1	AE2	MC	AE1 error	AE2 error	AE1 error rate	AE2 error rate
V-ii-2	3.19	2.70	2.73	0.46	−0.03	16.7	−1.3
V-ii-3	0.61	0.49	0.50	0.10	−0.01	20.9	−1.7
V-iii-1	8.30	6.82	6.97	1.32	−0.15	19.0	−2.2
V-iii-2	3.19	2.35	2.45	0.74	−0.10	30.1	−4.0
V-iii-3	0.61	0.41	0.44	0.17	−0.03	37.7	−6.2

Table 4 Discrete barrier option prices (case D)

	AE1	AE2	MC	AE1 error	AE2 error	AE1 error rate	AE2 error rate
I-i-1	8.30	7.92	7.94	0.36	−0.02	4.7	−0.3
I-i-2	3.19	2.98	2.99	0.20	−0.01	7.1	−0.5
I-i-3	0.61	0.56	0.56	0.04	−0.00	8.6	−0.8
I-ii-1	8.30	7.86	7.88	0.41	−0.02	5.5	−0.3
I-ii-2	3.19	2.94	2.96	0.23	−0.02	8.4	−0.6
I-ii-3	0.61	0.55	0.56	0.05	−0.00	10.3	−1.0
I-iii-1	8.30	7.80	7.82	0.47	−0.02	6.3	−0.3
I-iii-2	3.19	2.91	2.93	0.26	−0.02	9.7	−0.7
I-iii-3	0.61	0.54	0.55	0.06	−0.01	11.9	−1.1
II-i-1	8.30	8.01	8.03	0.27	−0.02	3.5	−0.2
II-i-2	3.19	3.03	3.04	0.15	−0.01	5.3	−0.4
II-i-3	0.61	0.57	0.57	0.03	−0.00	6.4	−0.6
II-ii-1	8.30	7.86	7.88	0.42	−0.02	5.6	−0.3
II-ii-2	3.19	2.94	2.96	0.23	−0.01	8.5	−0.5
II-ii-3	0.61	0.55	0.55	0.05	−0.00	10.3	−0.9
II-iii-1	8.30	7.71	7.74	0.56	−0.03	7.6	−0.4
II-iii-2	3.19	2.86	2.88	0.31	−0.02	11.7	−0.8
II-iii-3	0.61	0.53	0.54	0.07	−0.01	14.4	−1.3
III-i-1	8.30	8.16	8.18	0.12	−0.01	1.5	−0.2
III-i-2	3.19	3.11	3.12	0.07	−0.01	2.3	−0.3
III-i-3	0.61	0.59	0.59	0.01	−0.00	2.6	−0.4
III-ii-1	8.30	7.86	7.88	0.42	−0.02	5.6	−0.2
III-ii-2	3.19	2.94	2.95	0.23	−0.01	8.5	−0.5
III-ii-3	0.61	0.55	0.55	0.05	−0.00	10.5	−0.7
III-iii-1	8.30	7.56	7.60	0.70	−0.04	9.7	−0.6
III-iii-2	3.19	2.77	2.80	0.39	−0.03	15.1	−1.1
III-iii-3	0.61	0.51	0.52	0.09	−0.01	18.8	−1.7
IV-i-1	8.30	8.32	8.33	−0.03	−0.01	−0.4	−0.2
IV-i-2	3.19	3.20	3.21	−0.02	−0.01	−0.6	−0.2
IV-i-3	0.61	0.61	0.61	−0.01	−0.00	−0.9	−0.3
IV-ii-1	8.30	7.86	7.87	0.42	−0.02	5.6	−0.2

Table 4 continued

	AE1	AE2	MC	AE1 error	AE2 error	AE1 error rate	AE2 error rate
IV-ii-2	3.19	2.94	2.95	0.24	−0.01	8.7	−0.3
IV-ii-3	0.61	0.55	0.55	0.05	−0.00	10.8	−0.4
IV-iii-1	8.30	7.40	7.46	0.83	−0.06	11.7	−0.8
IV-iii-2	3.19	2.68	2.72	0.47	−0.04	18.5	−1.5
IV-iii-3	0.61	0.49	0.50	0.11	−0.01	23.2	−2.4
V-i-1	8.30	8.47	8.48	−0.19	−0.01	−2.3	−0.2
V-i-2	3.19	3.29	3.29	−0.10	−0.01	−3.4	−0.2
V-i-3	0.61	0.63	0.63	−0.02	−0.00	−4.3	−0.2
V-ii-1	8.30	7.86	7.87	0.42	−0.01	5.7	−0.1
V-ii-2	3.19	2.94	2.95	0.24	−0.00	8.8	−0.2
V-ii-3	0.61	0.55	0.55	0.06	−0.00	11.2	−0.1
V-iii-1	8.30	7.25	7.33	0.96	−0.08	13.8	−1.1
V-iii-2	3.19	2.60	2.65	0.54	−0.05	22.1	−2.0
V-iii-3	0.61	0.47	0.48	0.12	−0.01	27.9	−3.1

5 Conclusion

The paper applied a perturbation method for stochastic volatility models developed by Takahashi and Yamada (2009) to pricing path-dependent derivatives with discrete monitoring under stochastic volatility environment and obtained a concrete approximation formula for valuation of discrete barrier options. To our knowledge, this paper is the first one that shows an analytical approximation formula for pricing barrier options with discrete monitoring under stochastic volatility environment. Numerical experiments on double barrier options with discrete monitoring under Heston and λ-SABR models are also given.

Appendix A: Malliavin Calculus

Following Malliavin (1997) and Malliavin and Thalmaier (2006), this subsection summarizes basic facts on the Malliavin calculus which are necessary for this paper.

Let $(\mathcal{W}, \mu)$ be the d-dimensional Wiener space where

$$\mathcal{W} = \mathcal{W}^d = C_0([0, T] : \mathbf{R}^d) = \{w : [0, T] \to \mathbf{R}^d;\ \text{continuous},\ w(0) = 0\}$$

and μ is the Wiener measure. Next, let H be a Hilbert space such that

$$H = \left\{ h \in \mathcal{W}; h_i(t)(i = 1, \ldots, d) \text{ is absolute continuous with respect to } t \text{ and } \sum_{i=1}^{d} \int_0^T |\frac{dh_i(t)}{dt}|^2 dt < \infty \right\}$$

with an inner product $\langle h, h' \rangle_H = \sum_{i=1}^{d} \int_0^T \frac{dh_i(t)}{dt} \frac{dh_i'(t)}{dt} dt$. H is called the Cameron-Martin space.

Define $L^{\infty-}(\mathcal{W})$ as $L^{\infty-}(\mathcal{W}) = \cap_{p<+\infty} L^p(\mathcal{W})$ and a distance on $L^{\infty-}(\mathcal{W})$ as $d_{L^{\infty-}(\mathcal{W})}(F_1, F_2) = \sum_{j=1}^{\infty} 2^{-j} (\min\{\|F_1 - F_2\|_{L^j}, 1\})$, where $\|\cdot\|_{L^p}$ denotes the L^p-norm in $(\mathcal{W}, \mu)$. Let $L^p(\mathcal{W} : H)$ denote the space of measurable maps from $\mathcal{W}$ to H such that $\|f\|_H \in L^p(\mathcal{W})$. The same definition is made for $L^{\infty-}(\mathcal{W} : H)$.

Then, consider the space

$$\mathbf{D}_1^p(\mathcal{W} : G) = \Big\{ F \in L^p(\mathcal{W}, G) : \text{there exists } DF \in L^p(\mathcal{W} : H \otimes G) \\ \text{such that for } h \in H, \lim_{\epsilon \to 0} \frac{1}{\epsilon}[F(w + \epsilon h) - F(w)] = \langle DF, h \rangle_{H \otimes G} \Big\}.$$

Here, DF is called the (Malliavin) derivative of F. Due to the identification between the Hirbert space $L^p(\mathcal{W} : H \otimes G)$ and $L^p([0, T] \times \mathcal{W} : G)$, DF is a stochastic process $\{D_t F = (D_{t,1}F, \ldots, D_{t,d}F) : t \in [0, T]\}$ such that $\langle DF, h \rangle_{H \otimes G} = \sum_{i=1}^{d} \int_0^T (D_{t,i}F)(\frac{dh_i(t)}{dt}) dt$.

The norm of $\mathbf{D}_1^p(\mathcal{W} : G)$ is given by $\|F\|_{\mathbf{D}_1^p(\mathcal{W}:G)} = \|F\|_{L^p(\mathcal{W}:G)} + \|DF\|_{L^p(\mathcal{W}:H \otimes G)}$. Also, $\mathbf{D}_1^{\infty}(\mathcal{W} : G)$ is defined by $\mathbf{D}_1^{\infty}(\mathcal{W} : G) := \cap_{p<+\infty} \mathbf{D}_1^p(\mathcal{W} : G)$, and a distance on $\mathbf{D}_1^{\infty}(\mathcal{W} : G)$ is given by $d_{\mathbf{D}_1^{\infty}(\mathcal{W}:G)}(F_1, F_2) = \sum_{j=1}^{\infty} 2^{-j}$ $(\min\{\|F_1 - F_2\|_{\mathbf{D}_1^j(\mathcal{W}:G)}, 1\})$.

For $r \geq 2 (r \in \mathbf{N})$, we introduce the spaces:

$$\mathbf{D}_r^p(\mathcal{W} : G) = \{F \in \mathbf{D}_{r-1}^p(\mathcal{W} : G) : DF \in \mathbf{D}_{r-1}^p(\mathcal{W} : H \otimes G)\}$$

with $\|F\|_{\mathbf{D}_r^p(\mathcal{W}:G)} = \|F\|_{\mathbf{D}_{r-1}^p(\mathcal{W}:G)} + \|D^{r-1}F\|_{\mathbf{D}_1^p(H^{\otimes(n-1)} \otimes G)}$. We also define $\mathbf{D}_0^p(\mathcal{W} : G)$ as $\mathbf{D}_0^p(\mathcal{W} : G) = L^p(\mathcal{W} : G)$.

If $G = \mathbf{R}^n$, We denote $\mathbf{D}_r^p(\mathcal{W})$ as $\mathbf{D}_r^p(\mathcal{W} : G)$.

Some properties of these spaces are the following; $\mathbf{D}_{r'}^{p'}(\mathcal{W}) \subset \mathbf{D}_r^p(\mathcal{W})$, $r' \leq r$, and $p' \leq p$. The dual space of $(\mathbf{D}_r^q(\mathcal{W}))$, $(\mathbf{D}_r^q(\mathcal{W}))^*$ is given by $(\mathbf{D}_r^q(\mathcal{W}))^* = \mathbf{D}_{-r}^p(\mathcal{W})$, with $p^{-1} + q^{-1} = 1$.

Furthermore, define the space $\mathbf{D}_{\infty}(\mathcal{W}) = \cap_{p,r} \mathbf{D}_r^p(\mathcal{W})$. Then, $\mathbf{D}_{\infty}(\mathcal{W})$ is a complete metric space under a metric, $d_{\mathbf{D}^{\infty}(\mathcal{W})}(F_1, F_2) = \sum_{p,r=1}^{\infty} \eta_{p,r} (\min\{\|F_1 - F_2\|_{\mathbf{D}_r^p}, 1\})$ where $\eta_{p,r} > 0$ such that $\sum_{p,r}^{\infty} \eta_{p,r} < \infty$. Note that this topology on $\mathbf{D}_{\infty}(\mathcal{W})$ is independent of the choice of the sequence $\{\eta_{p,r}\}$. We call $F \in \mathbf{D}_{\infty}(\mathcal{W})$ the smooth functional in the sense of Malliavin.

Given $Z = (Z_1(w), \ldots, Z_d(w)) \in \mathbf{D}_1^p(\mathcal{W} : H)$, there exists $D_i^*(Z_i) \in L^p(\mathcal{W})$, $i = 1, \ldots, d$ such that $E[\int_0^T D_{t,i}F(w)Z_i(w)dt] = E[F(w)D_i^*(Z_i(w))]$ for all $F \in \mathbf{D}_1^{\infty}(\mathcal{W})$. Then, define $D^*Z := \sum_{i=1}^{d} D_i^*(Z_i(w))$. So, there exists $C_p > 0$ such that $\|D^*Z\|_{L^p} \leq C_p \|Z\|_{\mathbf{D}_1^p(\mathcal{W}:H)}$. We call D^*Z the divergence of Z.

Definition A.1 Let $F = (F_1, \ldots, F_n) \in \mathbf{D}_{\infty}(\mathcal{W} : \mathbf{R}^n)$ be the n-dimensional smooth functional, we call F a non-degenerate in the sense of Malliavin if the Malliavin covariance matrix $\{\sigma_F^{ij}\}_{1 \leq i,j \leq n}$

$$\sigma_F^{ij} := \langle DF_i, DF_j \rangle_H = \sum_{k=1}^{d} \int_0^T (D_{t,k} F_i(w))(D_{t,k} F_j(w))dt$$

is invertible $a.s.$ and

$$(\det \sigma_F)^{-1} \in L^{\infty-}(\mathcal{W}).$$

Theorem A.1 *Let $F \in \mathbf{D}_\infty(\mathcal{W} : \mathbf{R}^n)$ be a n-dimensional non-degenerate in the sense of Malliavin and $G \in \mathbf{D}_\infty(\mathcal{W})$. Then, for $\varphi \in C_b^1(\mathbf{R}^n)$,*

$$E[\partial_i \varphi(F) G] = E\left[\varphi(F) D^* \left(\sum_{j=1}^{n} G \gamma_{ij}^F DF^j\right)\right]$$

where $(\gamma_{ij}^F)_{1 \le i,j \le n}$ is the inverse matrix of Malliavin covariance of F.

Proof See Lemma III.5.2. of Malliavin (1997). □

Theorem A.2 *Let $F \in \mathbf{D}_\infty(\mathcal{W} : \mathbf{R}^n)$ be a non-degenerate functional. F has a smooth density $p^F \in \mathcal{S}(\mathbf{R}^n)$ where $\mathcal{S}(\mathbf{R}^n)$ denotes the space of all infinitely differentiable functions $f : \mathbf{R}^n \mapsto \mathbf{R}$ such that for any $k \ge 1$, and for any multi-index $\beta \in \{1, \dots, n\}^j$ one has $\sup_{x \in \mathbf{R}^n} |x|^k |\partial_\beta f(x)| < \infty$. (i.e. $\mathcal{S}(\mathbf{R}^n)$ is the Schwartz space and $\mathcal{S}'(\mathbf{R}^n)$ is its dual.)*

Proof See Theorem III.5.1. of Malliavin (1997). □

Definition A.2 Consider the space $\mathbf{D}_{-\infty}(\mathcal{W}) = \cup_{p,r} \mathbf{D}_{-r}^p(\mathcal{W})$, that is, the dual of $\mathbf{D}_\infty$. We call $F \in \mathbf{D}_{-\infty}(\mathcal{W})$ a distribution on the Wiener space. We define the duality form on $\mathbf{D}_{-\infty} \times \mathbf{D}_\infty$, $(F, G) \mapsto {}_{\mathbf{D}_{-\infty}}\langle F, G \rangle_{\mathbf{D}_\infty} = E[FG] \in \mathbf{R}$. We call this duality form the generalized expectation.

Let $F \in \mathbf{D}_\infty(\mathcal{W} : \mathbf{R}^n)$ be a non-degenerate functional and ν and p^F be the law and the smooth density of F, that is $\nu(dx) = \mu \circ F^{-1}(dx) = p^F(x)dx$ is the direct image by F of the Wiener measure on $\mathcal{W}$. We define the range O as $O := \{x : p^F(x) > 0\} \subset \mathbf{R}^n$.

By Malliavin (1997) and Malliavin and Thalmaier (2006) the conditional expectation of $g \in L^p(\mathcal{W}, \mu)$ conditioned by a set $\{w : F(w) = x\}$ in σ-field $\sigma(F)$, $E[g|F = x]$ gives a map,

$$E^F : L^p(\mathcal{W}, \mu) \ni g \mapsto E[g|F = x] \in L^p(O, \nu). \tag{A.2}$$

Watanabe (1983, 1984) introduced the distribution on Wiener space as composition of a non-degenerate map F by a Schwartz distribution T. The next theorem restates the result of Watanabe (1984) in terms of Malliavin (1997) and Malliavin and Thalmaier (2006).

Theorem A.3 [Watanabe 1984] *Let* $F \in \mathbf{D}_\infty(\mathcal{W} : \mathbf{R}^n)$ *be a non-degenerate functional. Let* p^F *be the density of* F *and* $O := \{x : p^F(x) > 0\} \subset \mathbf{R}^n$.

1. *Let* $\mathcal{S}'(\mathbf{R}^n)$ *be the Schwartz distributions. The map* $\mathcal{S}(\mathbf{R}^n) \ni T \mapsto T \circ F \in \mathbf{D}_\infty$ *can be uniquely extended to a map:*

$$(E^F)^* : \mathcal{S}'(\mathbf{R}^n) \ni T \mapsto T \circ F \in \tilde{\mathbf{D}}_{-\infty} := \cup_{s \geq 0} \cap_{q \geq 1} \mathbf{D}^q_{-s} \subset \mathbf{D}_{-\infty}. \quad \text{(A.2)}$$

   $(E^F)^*$ *is called the lifting up of* T.
2. *The conditional expectation defines a map*

$$E^F : \mathbf{D}_\infty \ni G \mapsto E^F[G] \in \mathcal{S}(O), \quad \text{(A.2)}$$

   where $\mathcal{S}(O)$ *stands for the Schwartz space of the rapidly decreasing functions on* $O = \{x : p^F(x) > 0\} \subset \mathbf{R}^n$. *We call this map the push down of* G.
3. *The following duality formula is obtained :*

$${}_{\mathbf{D}_{-\infty}}\langle (E^F)^* T, G \rangle_{\mathbf{D}_\infty} = \langle T, E^F[G] \rangle_{p^F(x)dx}, \quad \text{(A.2)}$$

   where the notation $\langle \cdot, \cdot \rangle_{p^F(x)dx}$ *is denoted as follows:*

$$\langle T, E^F[G] \rangle_{p^F(x)dx} = {}_{\mathcal{S}'}\langle T, E^F[G] p^F \rangle_{\mathcal{S}}. \quad \text{(A.2)}$$

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